WEAK TURBULENCE OF CAPILLARY WAVES

V. E. Zakharov and N. N. Filonenko

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 8, No. 5, pp. 62–67, 1967

In recent years the theory of weak turbulence, i.e. the stochastic theory of nonlinear waves [1, 2], has been intensively developed. In the theory of weak turbulence nonlinearity of waves is assumed to be small; this enables us, using the hypothesis of the random nature of the phases of individual waves, to obtain the kinetic equation for the mean squares of the wave aplitudes.

In many cases of weak turbulence a situation arises where damping is considerable in the region of large wave numbers and is separated from the region where the basic energy of the waves is concentrated (as a result either of pumping or of the initial conditions) with a wide region of transparency. In [3, 4] the hypothesis was stated that weak turbulence in these cases is completely analogous to hydrodynamic turbulence for large Reynolds numbers in the sense that in the region of transparency a universal spectrum is established which is determined only by the flow of energy into the region of large wave numbers. The spectrum of hydrodynamic turbulence $\varepsilon_k \sim k^{-5/3}$ was obtained by A. N. Kolmogorov and A. M. Obukhov [5, 6] from dimensional considerations. In the case of weak turbulence the spectrum is obtained as an exact solution of the stationary kinetic equation.

Below the case of weak turbulence of capillary waves on the surface of a liquid is considered.

A kinetic equation is obtained for capillary waves. It is significant that in this case the basic contribution to interaction is provided by the process of the decomposition of a wave into two and by the process of two waves merging into one.

It is shown that the collision term of the kinetic equation vanishes with the solution $\varepsilon_k \sim k^{-7/4}$. Arguments are advanced in favor of the fact that this solution can be interpreted as a universal spectrum in the region of transparency.

1. The kinetic equation. As is known, the law of wave dispersion on the surface of an infinitely deep liquid has the form $\omega_k = (\alpha k^3 + gk)^{1/2}$, where α is the coefficient of surface tension. The density of the liquid is taken as equal to unity.

The case $k \gg g/\alpha$ is considered. Here the effect of gravitational forces can be neglected, and the spectrum assumes the form $\omega_k = (\alpha k^3)^{1/2}$.

Vibrations of the surface of a liquid, without viscosity taken into account, are described by the following system of equations (subsequently viscosity will be taken into account phenomenologically):

$$\Delta\Phi = 0, \quad z < \eta, \quad \eta_t - \Phi_z = -\eta_x\Phi_x - \eta_y\Phi_y|_{z=\eta},$$
$$\Phi_t - \alpha(\eta_{xx} + \eta_{yy}) = {}^1/_2(\nabla\Phi)^2|_{z=\eta},$$
$$\Phi(x, y, z, t)|_{z=-\infty} = 0. \qquad (1.1)$$

Here $\Phi(x, y, z, t)$ is the velocity potential; $\eta(x, y, t)$ is the deviation of the surface from equilibrium. The z axis is directed away from the liquid. Without loss of generality we can set the pressure equal to zero. This equation has an integral of motion: the energy of the waves, which with an accuracy to terms of the third order with respect to $\eta(x, y, t)$ has the form

$$\varepsilon = \frac{1}{2}\int \alpha[\eta_x^2 + \eta_y^2]\,dx\,dy + \frac{1}{2}\int dx\,dy \int_{-\infty}^{\eta} (\nabla\Phi)^2\,dz.$$

Let us turn to the Fourier transforms of x and y in Eqs. (1.1), using the Laplace equation and the boundary condition. Here we make use of the smallness of nonlinearity, retaining in the Fourier series terms up to the second order of smallness with respect to the amplitude of vibrations

$$\frac{\partial\eta_k}{\partial t} - k\Psi_k = \int [(\mathbf{k}\mathbf{k}_1) - |\mathbf{k}|\,|\mathbf{k}_1|]\,\Psi_{k_1}\eta_{k_2}\delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}\,d\mathbf{k}_1\,d\mathbf{k}_2,$$
$$\frac{\partial\Psi_k}{\partial t} + \alpha k^2\eta_k = -\frac{1}{2}\int [(\mathbf{k}_1\mathbf{k}_2) + |\mathbf{k}_1|\,|\mathbf{k}_2|]\,\Psi_{\mathbf{k}_1}\Psi_{\mathbf{k}_2}\delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}\,d\mathbf{k}_1\,d\mathbf{k}_2,$$
$$\Psi(x, y, t) = \Phi(x, y, z, t)|_{z=\eta}. \qquad (1.2)$$

It is convenient to turn to the new variables a_k and a^*_k, the complex amplitudes of waves

$$\eta_k = (4/\alpha k)^{1/4}(a_k + a_{-k}^*),$$
$$\Psi_k = -i(4\alpha k)^{1/4}(a_k - a_k^*).$$

Here a_k, a_{-k}^* are normed in such a way that

$$\varepsilon^{(0)} = \int \omega_k |a_k|^2\,d\mathbf{k} \qquad (1.3)$$

where ε_0 is the quadratic part of the wave energy.

The equation for capillary waves in terms of these variables has the form

$$\frac{\partial a_k}{\partial t} - i\omega_k a_k = i\int V_{kk_1k_2}a_{k_1}a_{k_2}\delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}\,d\mathbf{k}_1\,d\mathbf{k}_2 + 2i\int V_{k_1kk_2}a_{k_1}a_{k_2}^*\delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}\,d\mathbf{k}_1\,d\mathbf{k}_2 + i\int U_{kk_1k_2}a_{k_1}^*a_{k_2}^*\delta_{\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2}\,d\mathbf{k}_1\,d\mathbf{k}_2, \qquad (1.4)$$

$$V_{kk_1k_2} = \left(\frac{\alpha}{4\sqrt{2}}\right)^{1/4}\left\{\left(\frac{|\mathbf{k}|\,|\mathbf{k}_1|}{|\mathbf{k}_2|}\right)^{1/4}[(\mathbf{k}-\mathbf{k}_1)^2 - \mathbf{k}_2^2] + \left(\frac{|\mathbf{k}|\,|\mathbf{k}_2|}{|\mathbf{k}_1|}\right)^{1/4}[(\mathbf{k}-\mathbf{k}_2)^2 - \mathbf{k}_1^2] - \left(\frac{|\mathbf{k}_1|\,|\mathbf{k}_2|}{|\mathbf{k}|}\right)^{1/4}[(\mathbf{k}_1-\mathbf{k}_2)^2 - \mathbf{k}^2]\right\},$$

$$U_{kk_1k_2} = \left(\frac{\alpha}{4\sqrt{2}}\right)^{1/4}\left\{\left(\frac{|\mathbf{k}|\,|\mathbf{k}_1|}{|\mathbf{k}_2|}\right)^{1/4}[\mathbf{k}_2^2 - (\mathbf{k}-\mathbf{k}_1)^2] + \left(\frac{|\mathbf{k}|\,|\mathbf{k}_2|}{|\mathbf{k}_1|}\right)^{1/4}[\mathbf{k}_1^2 - (\mathbf{k}-\mathbf{k}_2)^2] + \left(\frac{|\mathbf{k}_1|\,|\mathbf{k}_2|}{|\mathbf{k}|}\right)^{1/4}[\mathbf{k}^2 - (\mathbf{k}_1-\mathbf{k}_2)^2]\right\}. \qquad (1.5)$$

We note that $V_{kk_1k_2}$ and $U_{kk_1k_2}$ are homogeneous functions of degree 9/4, satisfying the symmetry conditions

$$V_{kk_1k_2} = V_{kk_2k_1}, \qquad U_{kk_1k_2} = U_{kk_2k_1} = U_{k_2kk_1}$$

The functions V and U depend only on the moduli of their arguments. We note that capillary waves exhibit a "split law of dispersion" [7], i.e., the conditions

$$\omega_k = \omega_{k_1} + \omega_{k_2}, \qquad \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$$

can be satisfied simultaneously.

It follows from this fact that a monochromatic capillary wave with the wave vector **k** is unstable relative to simultaneous excitation of a wave pair with the wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$ (split instability).

Let us proceed to a statistical description of the vibrations. We assume that the system of waves is statistically homogeneous and, furthermore, that the phases of the individual vibrations are completely chaotic. In accordance with the established terminology [1, 2] we call such a condition weak turbulence of waves. To describe turbulence, we can obtain a kinetic equation for $n_k = |a_k|^2$ in the manner of A. A. Galeev and V. I. Karpman [8]

$$\partial n_k / dt = St(n, n) - 2\nu k^2 n_k, \tag{1.6}$$

$$St(n, n) = 4\pi \int |V_{kk_1k_2}|^2 (n_{k_1} n_{k_2} - n_k n_{k_1} - n_k n_{k_2}) \times \times \delta_{k-k_1-k_2} \delta_{\omega_k - \omega_{k_1} - \omega_{k_2}} d\mathbf{k}_1 d\mathbf{k}_2 + + 8\pi \int |V_{k_1kk_2}|^2 (n_{k_1} n_{k} + + n_k n_{k_1} - n_k n_{k_2}) \delta_{\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2} \delta_{\omega_k - \omega_{k_1} + \omega_{k_2}} d\mathbf{k}_1 d\mathbf{k}_2. \tag{1.7}$$

The term $2\nu k^2 n_k$ where ν is the coefficient of viscosity, is introduced into the kinetic equation. This term describes the viscous damping of waves [9].

According to formula (1.3), n_k is associated with the spectral energy density by the relation $\varepsilon_k = \omega_k n_k$. The quantity n_k can be interpreted as the density of the wave number in k-space [1–4].

Equation (1.7) exhibits the law of the conservation of energy

$$\frac{\partial}{\partial t} \int \omega_k n_k \, d\mathbf{k} + 2 \int \nu k^2 \omega_k n_k \, d\mathbf{k} = 0. \tag{1.8}$$

2. **Solution of the kinetic equation.** Let us consider the equation

$$St(n, n) = 0\,. \tag{2.1}$$

We shall seek cylindrically symmetric solutions of this equation. We make use of the fact that the coefficient function is independent of the angles, and carry out an averaging process with respect to the angles in Eq. (2.1). For this we represent the δ-function of the wave vectors in the form

$$\delta_{k \pm k_1 \pm k_2} = \int e^{i(\mathbf{r}, k \pm k_1 \pm k_2)} d\mathbf{r}\,.$$

Having integrated Eq. (2.1) with respect to the angles between the vectors **r** and **k**, **r** and $\mathbf{k}_1$, **r** and $\mathbf{k}_2$, we proceed to integrate the vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ with respect to the moduli, where the δ-function of the wave numbers is replaced by the expression

$$\int_0^\infty J_0(kr) J_0(k_1 r) J_0(k_2 r)\, r\, dr = \frac{1}{\Delta},$$

$$\Delta = {}^1/_2 \sqrt{2 [k_1^2 k_2^2 + k^2 k_1^2 + k^2 k_2^2 - k^4 - k_1^4 - k_2^4}\,.$$

Here Δ is the area of the triangle formed by the vectors **k**, $\mathbf{k}_1$ and $\mathbf{k}_2$.

After this in Eq. (2.1) we go over to the variables

$$\omega = k^{3/2}, \qquad \omega_1 = k_1^{3/2}, \qquad \omega_2 = k_2^{3/2}$$

and to preserve symmetry of the kernel we multiply the equation by the quantity $\omega^{1/3}$.

After integrating with respect to the variable ω_2 we obtain

$$\int_0^\omega P_{\omega, \omega_1, \omega-\omega_1} (n_{\omega_1} n_{\omega-\omega_1} - n_\omega n_{\omega_1} - n_\omega n_{\omega-\omega_1})\, d\omega_1 + + 2\int_0^\infty P_{\omega+\omega_1, \omega, \omega_1} (n_{\omega_1} n_{\omega+\omega_1} + n_\omega n_{\omega+\omega_1} - n_\omega n_{\omega_1})\, d\omega_1 = 0,$$

$$P_{\omega\omega_1\omega_2} = \left((\omega\omega_1\omega_2)^{1/3} |V_{\omega, \omega_1, \omega_2}|^2\right) \times \times \left(2 [\omega_1^{4/3} (\omega - \omega_1)^{4/3} + (\omega\omega_1)^{4/3} + + \omega^{4/3} (\omega_1 - \omega)^{4/3} - \omega^{8/3} - \omega_1^{8/3} - (\omega - \omega_1)^{8/3}]\right)^{-1/2} \tag{2.2}$$

Here $P_{\omega, \omega_1, \omega_2}$ is a homogeneous positive definite function of degree 8/3

$$P_{\omega, \omega_1, \omega-\omega_1} \sim P_{\omega+\omega_1, \omega, \omega_1} \sim \omega_1^2 \omega^{2/3} \quad \text{for } \omega_1 \ll \omega\,. \tag{2.3}$$

We shall seek the solution of Eq. (2.2) in the form $n_\omega = A\omega^s$, where A is an arbitrary constant, while s is an unknown quantity.

We carry out substitution of the variables in the second integral of Eq. (2.2) in accordance with the formulas

$$\omega_1 \to \frac{\omega - \omega_1}{\omega_1} \omega_1, \qquad d\omega_1 \to -\left(\frac{\omega}{\omega_1}\right)^2 d\omega_1\,.$$

The function, owing to its homogeneity and symmetry (see (1.6)) is transformed as follows:

$$P_{\omega+\omega_1, \omega_1, \omega} \to P_{\omega\omega/\omega_1, (\omega-\omega_1)\omega/\omega_1, \omega_1\omega/\omega_1} \to \to (\omega/\omega_1)^{8/3} P_{\omega, \omega-\omega_1, \omega_1}$$

It is now seen that after such a substitution two integrals are turned into one, while the integrand is easily factorized.

The equation for the unknown quantity s thus has the form

$$\int_0^\omega d\omega_1 \frac{P_{\omega, \omega_1, \omega-\omega_1} [\omega_1^s (\omega - \omega_1)^s - \omega^s \omega_1^s - \omega^s (\omega - \omega_1)^s]}{\omega_1^{2s+14/3} (\omega - \omega_1)^{2s+14/3}} \times \times \int_0^\omega [\omega_1^{2s+14/3} (\omega - \omega_1)^{2s+14/3} - \omega^{2s+14/3} \omega_1^{2s+14/3} - - \omega^{2s+14/3} (\omega - \omega_1)^{2s+14/3}]\, d\omega_1 = 0.$$

It is obvious that the integrand vanishes for values of s equal to −1 and −17/6.

Owing to the positive definiteness of the function P, Eq. (2.1) has no solutions of other powers.

The solution $n_\omega^{(1)} = \text{const}/\omega$, i.e., the Rayleigh-Jeans distribution, corresponds to the first of the equations in (2.2).

The solution $n_\omega^{(2)} = \text{const}/\omega^{17/16}$ corresponds to the second root.

In k-space the distributions

$$n_k^{(1)} = \text{const}\, k^{-3/2}, \qquad n_k^{(2)} = \text{const}\, k^{-17/4}$$

correspond to these solutions; in cylindrical normalization this gives

$$\varepsilon_k^{(1)} = \text{const}\, k, \qquad \varepsilon_k^{(2)} = \text{const}\, k^{-7/4}$$

for the spectral density.

For this solution to have a physical meaning, it is necessary for the integrals in Eq. (2.7) to converge. Let us first consider the convergence in the region of small k.

We note that in the first term of Eq. (2.2) integration over the region $\omega - \omega_1 \ll \omega$ gives the same contribution as integration over the region $\omega_1 \ll \omega$. Taking this into consideration, we collect all the terms passing to infinity as $\omega_1 \to 0$. We obtain

$$2\int\{P_{\omega_1,\,\omega_1,\,\omega-\omega_1}[n_\omega n_{\omega-\omega_1} - n_\omega n_{\omega_1}] + P_{\omega+\omega_1,\omega,\,\omega_1}[n_{\omega_1}n_{\omega+\omega_1} - n_\omega n_{\omega_1}]\}\,d\omega_1\,. \tag{2.4}$$

Taking into account the asymptotic property of (2.3), these terms have the order

$$\frac{1}{\omega^{1/3}}\frac{\partial n_\omega}{\partial\omega}\int n_{\omega_1}\omega_1^4\,d\omega_1\,. \tag{2.5}$$

Hence it follows that integrals in Eq. (2.2) converge for both of the solutions obtained here as $\omega_1 \to 0$. Let us consider the convergence as $\omega_1 \to \infty$.

In this case the terms

$$2n_\omega\int P_{\omega+\omega_1,\,\omega,\,\omega_1}(n_{\omega+\omega_1} - n_\omega)\,\omega^3 n_\omega\,d\omega_1 \sim \omega^3 n_\omega\int \omega_1^{2/3}\frac{\partial n_{\omega_1}}{\partial\omega_1}\,d\omega_1 \tag{2.6}$$

are the most critical.

It is obvious that in this limiting case the integrals in the equation converge for both solutions.

3. **A physical interpretation of the solutions.** Let us consider the problem concerned with the damping of capillary waves. We estimate the orders of various terms in Eq. (1.7). Let τ be the characteristic damping time. The term $\partial n/\partial t$ has the order n/τ; the term St(n, n) then has the order $\nu^2 n^2 k^2/\omega_k \sim n^2/k^5$ and for sufficiently large k it is much larger than the term $\partial n/\partial t$. Thus the term $\partial n/\partial t$ is significant only for small k.

We denote the influence boundary of the term $\partial n/\partial t$ by a. Furthermore, it is clear that viscosity has an effect only for sufficiently large k. We denote the viscosity influence boundary by b. We consider the case $b \gg a$. We attempt to approximate the solution of Eq. (1.7) in the region $a \ll k \ll b$ with the aid of the exact solutions of Eq. (2.1). Let us first consider the Rayleigh-Jeans distribution.

Owing to the convergence of the integrals in Eq. (2.1), for the Rayleigh-Jeans distribution the principal contribution to the integral is determined by the region $k_1 \sim k$ and has the order T^2k^2. On the other hand, the term $\nu k^2 n_k$ has the order $\nu T k^{1/2}$. Hence we see that viscosity cannot lead to cutoff of the Rayleigh-Jeans distribution for large k and $b = \infty$. But since the total energy for the Rayleigh-Jeans distribution diverges for large k, this means that the Rayleigh-Jeans solution cannot be realized in the given problem.

Let us now consider the solution $n_k = ck^{-17/4}$. The order of the collision term for this solution is $c^2k^{-7/2}$, while the order of the viscous term is $\nu ck^{-9/4}$. Hence the boundary of influence for the viscous term is $b \sim (c/\nu)^{4/5}$.

It is natural to expect that the solution is rapidly damped for $k > b$.

The solution $n_k = ck^{-17/4}$ rapidly diminishes for $k \gg a$; therefore the principal part of the energy is included in the region $k \sim a$, where nonsteadiness is significant. Let the solution in this region have the order n_0. From the joining condition on the boundary of the region containing the energy we have

$$n_0 \sim ca^{-17/4}.$$

Thus the true solution differs considerably from the solution $n_k = ck^{-17/4}$ in the regions $k \lesssim a$ and $k \gtrsim b$. Integration in the collision term is carried out over the entire space of wave numbers including these regions. Here the contribution of the region $a \ll k \ll b$ has the order $c^2k^{-7/2}(a/k)^{13/4}$.

From formulas (2.5) and (2.6) we can estimate the contributions of the regions $k \lesssim a$ and $k \gtrsim b$. They equal $c^2k^{-7/2}(a/k)^{13/4}$ and $c^2k^{-7/2}\cdot(k/b)^{11/4}$, respectively. It is obvious that for $a \ll k \ll b$ these contributions are negligibly small.

Let us now calculate the quantity of energy dissipated per unit time. This quantity is given by formula (1.8). The principal contribution to the integral is determined by the upper limit

$$p \sim c\int^{b}\frac{\nu k^2\omega_k}{k^{17/4}}\,kdk \sim \alpha^{1/2}c^2. \tag{3.2}$$

We find that the quantity of energy dissipated or, what is the same, the energy flow into the region of large k does not depend on the value of the viscosity coefficient. The solution in the region $a \ll k \ll b$ can be rewritten to the form

$$n_k \sim p^{1/2}\alpha^{-1/4}k^{-17/4}. \tag{3.3}$$

Hence

$$n_0 \sim p^{1/2}\alpha^{-1/4}a^{-17/4}. \tag{3.4}$$

From the kinetic equation (1.7) we can establish that the energy flux p is proportional to n^2, i.e., $n \sim p^{1/2}$. It is easy to see that $n_k \sim p^{1/2}\alpha^{-1/4}k^{-17/4}$ is the only power function which satisfies this condition with respect to dimensionality. The total wave energy has the order

$$\varepsilon \sim \omega a^2 n_0 \sim \alpha^{1/2}a^{7/2}n_0. \tag{3.5}$$

From the law of the conservation of energy we have

$$\varepsilon/\tau \sim p\,. \tag{3.6}$$

From formulas (3.3)-(3.6) we find that

$$1/\tau \sim n_0 a^5 \sim n_0\varepsilon/\alpha\,. \tag{3.7}$$

Knowing τ, it is easy to establish that the terms $\partial n/\partial t$ and St(n, n) are indeed comparable for $k \sim a$. We note that the kinetic equation is applicable only for small nonlinearity, when $\varepsilon/\alpha \ll 1$. Formula (3.6) can be rewritten to the form $n_0/\tau \sim \partial n_0/\partial t \sim n_0^2 a^5$.

Hence $n \sim n_0\tau/t$, i.e., n decreases in inverse proportion to time. Let us find the boundary of viscous damping

$$b \sim (c/\nu)^{4/5} \sim n_0^{17/5}/\nu^{4/5}\,.$$

The criterion of existence for the region $a \ll k \ll b$ leads to the condition $b \gg a$, which, as can be easily confirmed, coincides with the condition $\tau\nu a^2/ \gg 1$. This is to say, it coincides with the condition that the decrement of the nonlinear damping is much larger than the decrement of viscous damping. Hence we obtain the final criterion of theory applicability $\nu a^2/\omega_0 \ll \varepsilon/\alpha \ll 1$.

The constructed pattern of weak turbulence in capillary waves has much in common with the pattern of turbulence for an incompressible liquid in the case of large Reynolds numbers. In both cases the wave-number space can be divided into three regions: the region containing energy, the intermediate (inertial) region, and the damping region. At the same time the spectrum of energy in the region containing energy and in the intermediate region does not depend on the coefficient of viscosity (the coefficient of viscosity determines only the upper boundary of the intermediate region). In both cases the spectrum of energy in the intermediate region is determined by only a single quantity, the energy flow from the region containing energy. For hydrodynamic turbulence this enables us to find this spectrum $\varepsilon_k \sim k^{-5/3}$, using dimensional considerations. Turbulence of capillary waves contains the additional dimensional parameter α. This does not allow us directly to use the dimensional considerations, but after obtaining the kinetic equation we can establish that the energy flow is proportional to the square of the wave energy. This allows us to construct the expression $\varepsilon_k \sim \sim \alpha^{-1/4} p^{1/2} k^{-17/4}$ which turns out to be the exact solution of the kinetic equation.

In the theory of hydrodynamic turbulence the pattern described above is based on the hypothesis of the local character of turbulence, i.e., on the assumption that only dimensions of the same order interact intensively with one another. This hypothesis for capillary waves is factually confirmed by relations (3.2). We further note that the problem of the stability of the turbulence pattern set up here remains unresolved.

In conclusion, the authors thank R. Z. Sagdeev for his contribution to the work.

REFERENCES

1. B. B. Kadomtsev, "Turbulence of plasmas," collection: Problems of the Theory of Plasmas [in Russian], Atomizdat, no. 4, p. 188, 1964.

2. A. A. Galeev, V. I. Karpman, and R. Z. Sagdeev, "Multiparticle aspects of the theory of turbulence of a plasma," Nuclear Fusion, vol. 5, no. 1, p. 20, 1965.

3. V. E. Zakharov, "Weak turbulence in media with a decay spectrum," PMTF [Journal of Applied Mechanics and Technical Physics], no. 4, p. 35, 1965.

4. V. E. Zakharov and N. N. Filonenko, "Energy spectrum of stochastic gravitational waves," Dokl. AN SSSR, vol. 170, no. 6, p. 1292, 1966.

5. A. M. Kolmogorov, "The local structure of turbulence in an incompressible liquid for very large Reynolds numbers," Dokl. AN SSSR, vol. 30, no. 4, p. 299, 1941.

6. A. M. Obukhov, "On energy distribution in the spectrum of a turbulent flow," Izv. AN SSSR, Ser. geogr. i fiz., vol. 5, no. 4, 5, p. 453, 1941.

7. A. A. Vedenov, Introduction to the theory of a weakly turbulent plasma [in Russian], Atomizdat, no. 3, p. 203, 1963.

8. A. A. Galeev and V. I. Karpman, "A turbulent theory of a slightly unbalanced rarefied plasma," Zh. eksperim. i teor. fiz., vol. 44, no. 2, p. 592, 1963.

9. L. D. Landau and E. M. Lifshits, Mechanics of Solids [in Russian], Gostekhizdat, 1953.

7 February 1967 Novosibirsk